Time Asymmetry in Quantum Physics - I. Theoretical Conclusion from Resonance and Decay-Phenomenology

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Abstract

It is explained how the unification of resonance and decay phenomena into a consistent mathematical theory leads to quantum mechanical time-asymmetry. This provides the theoretical basis for a subsequent paper II in which the interpretation and experimental demonstration of this time-asymmetry is discussed.

1 Introduction

Within the framework of traditional quantum theory one does not have a consistent theory of resonance and decay phenomena. One has various empirical concepts and useful methods, but, many puzzles, questions and contradictions remain. In non-relativistic physics, one has at least a generally accepted calculational scheme, the Weisskopf-Wigner approximation [1].

In the relativistic domain, one cannot even agree upon an *approximate* description of resonances. One is not sure whether it makes sense at all to describe resonances as separate entities which can be characterized by two well defined quantities, the mass and the width. In particular, recently in connection with the Z- and W-bosons it was mentioned that the definition of M and Γ is just a convention and as long as this was done consistently,

more or less any parametrization of the complex pole position was acceptable. With this argument one then justifies the use of the old parametrization of the complex pole position in terms of the non-gauge invariant on-the-mass-shell mass M_Z and width Γ_Z [2].

In non-relativistic quantum mechanics, on the basis of the Weisskopf-Wigner approximation [1] the width Γ of a Breit-Wigner energy distribution (2.2) is connected to the inverse lifetime τ. The Weisskopf-Wigner approximate methods provide only a vague and approximate relation [3], see (2.9) below. If $\Gamma = \frac{\hbar}{\tau}$ could be established as an exact relation, then one could use this relation as the criterion for the right definition of the width Γ. With the width precisely defined, this would then also define the mass M and therewith uniquely fix the two parameters (M, Γ) of a relativistic resonance. The first task, therefore, is to obtain a consistent quantum theory that unifies resonance scattering and decay phenomena, such that a relation $\Gamma = f(\tau)$, preferably, $\Gamma = \frac{\hbar}{\tau}$ is obtained as a result of this theory.

Such a theory cannot be obtained within the framework of conventional quantum mechanics using the Hilbert space axiom, because – as is well known – the Hilbert space does not contain a vector (or a state) with exponential time evolution. To obtain exponential Born probabilities which are needed to define the lifetime τ of exponential decay, one has to use generalized vectors, like e.g., Dirac kets, but still more generalized. Whereas Dirac kets $|E\rangle$ are eigenkets (i.e., continuous functionals) with eigenvalues from the continuous energy spectrum, these new generalized eigenvectors are eigenkets with complex eigenvalue $|E_R - i\Gamma/2^-\rangle$; for this reason, we call them Gamow kets. In mathematical terms, they are continuous functionals on a Hardy space Φ_-, whereas a Dirac ket, if defined at all, is mathematically defined as a continuous functional on a Schwartz space, *cf.* [4].

To obtain these Gamow kets with Breit-Wigner width Γ and lifetime $\tau = \frac{\hbar}{\Gamma}$, we have to modify just one of the traditional axioms of quantum mechanics, the Hilbert space axiom, and replace it with the Hardy space axiom. The Hardy space axiom mathematically distinguishes between prepared in-states and detected out-observables (usually miss-named out-states). Describing the set of in-states by $\Phi_- \subset \mathcal{H}$ and the set of detected out-observables by $\Phi_+ \subset \mathcal{H}$, both dense in the same Hilbert space $\mathcal{H}$ for a particular quantum system, one obtains a consistent and exact theory that unifies quantum resonances and decay: From the S-matrix pole definition of a resonance, one obtains a Gamow ket (functional on Φ_-) with Breit-Wigner (Lorentzian) energy distribution and exponential time evolution. This Gamow ket represents

the resonance per se (without background).

But, one also predicts as a mathematical consequence of the new Hardy space boundary conditions, $\phi^+(t) \in \Phi_-$ for the Schrödinger equation and $\psi^-(t) \in \Phi_+$ for the Heisenberg equation, a time-asymmetric semigroup evolution. The semigroup is in contrast to the reversible unitary group evolution of Hilbert space quantum mechanics. The semigroup evolution introduces a new concept into quantum mechanics, the semigroup time, $t_0 = 0$, which does not exist in conventional quantum mechanics, where the time extends over $-\infty < t < \infty$. The quantum mechanical "beginning of time" $t_0(> -\infty)$ and the experimental demonstration of it will be discussed in a subsequent paper. As a preparation of this, we explain in the present paper why quantum mechanical time asymmetry $t_0 = 0 < t < \infty$ is a consequence of the exact equality $\tau = \frac{\hbar}{\Gamma}$.

2 Resonances and Decaying States

Resonances and decaying states are widely believed to be different manifestations of the same entities.

Resonances $\mathcal{R}$ appear as intermediate states of scattering processes

$$1 + 2 \to \mathcal{R} \to 3 + 4, \quad \text{for example} \quad e^+e^- \to Z^0 \to \mu^+\mu^-\,, \tag{2.1}$$

when the scattering cross section of angular momentum j, $|a_j^{BW}(E)|^2$ is described by a Breit-Wigner energy distribution, (also called Lorentzian):

$$a_j^{BW} = \frac{r_\eta}{E - (E_R - i\frac{\Gamma}{2})}; \quad 0 \le E < \infty\,. \tag{2.2}$$

Resonances are thus characterized by the angular momentum j and by the resonance energy E_R (or resonance mass M in the relativistic case) and the resonance width Γ.

Usually the Lorentzian (2.2) is not sufficient to fit the experimental cross section (and asymmetry data) for processes like (2.1); in addition to the resonance amplitude (2.2) there is always a background amplitude B_j and the partial cross section of angular momentum j is fitted to:

$$\sigma_j(E) \sim |a_j(E)|^2 = |a_j^{BW}(E) + B_j(E)|^2\,. \tag{2.3}$$

Decaying states $\phi^D(t)$ are considered as the starting points of an exponential time evolution. Decaying states are observed in processes like

$$D \to \eta_1, \eta_2, \cdots, \quad \text{e.g., } K_S^0 \to \pi^+\pi^-, \pi^0\pi^0 \tag{2.4}$$

where $\eta_1, \eta_2, \cdots$ denote different decay products (or decay channels). The decay products, or more precisely the properties of the decay products η are described by the out "state" vector ψ_η, or out-observable $\Lambda_\eta = |\psi_\eta\rangle\langle\psi_\eta|$.

The decaying state ϕ^D is characterized by $(E_D, 1/\tau \equiv R)$ (or $(M, 1/\tau \equiv R)$) where τ is the lifetime (for the relativistic case, in the rest frame) and R is the total initial decay rate. The lifetime τ is measured by fitting the experimental counting rate, $\frac{1}{N}\frac{\Delta N_\eta(t)}{\Delta t}$, for any decay product η to the partial decay rate $R_\eta(t)$ (the intensity of the η emission as a function of time), for which one assumes the empirical exponential law

$$\frac{1}{N}\frac{\Delta N_\eta(t_i)}{\Delta t_i} \approx R_\eta(t) = R_\eta(0)e^{-t/\tau} = R_\eta(0)e^{-Rt}\,. \tag{2.5}$$

Here $R(t) = \sum_\eta R_\eta(t)$ is the total decay rate and $R = R(0)$ is the total initial decay rate; $\Delta N_\eta(t_i)$ is the number of the decay products η registered by the η-detector during the time interval Δt_i around t_i.

The theoretical decay rates $R_\eta(t)$ and the probabilities $P_\eta(t)$, are according to a fundamental axiom of quantum mechanics, given by the quantum mechanical Born probabilities of the observable Λ_η in the (decaying) state $\phi^D(t)$:

$$P_\eta(t) = Tr(\Lambda_\eta|\phi^D(t)\rangle\langle\phi^D(t)|) = |\langle\psi_\eta|\phi^D(t)\rangle|^2 \tag{2.6}$$

where $\Lambda_\eta = |\psi_\eta\rangle\langle\psi_\eta|$ is the projection operator on the subspace of properties of the decay products which are registered by the η-detector. The partial decay rates (also misleadingly called partial widths when multiplied by $\hbar$, $\Gamma_\eta = \hbar R_\eta(0)$) are the time derivatives of the probabilities $P_\eta(t)$.

$$R_\eta(t) = \frac{d}{dt}P_\eta(t). \tag{2.7}$$

The experimental definition of the lifetime τ, given by (2.5) and the relation between the total initial decay rate and the lifetime τ, $R = \frac{1}{\tau}$, is based on the validity of the exponential law (2.5) for the rate $R(t)$, and therefore also on the exponential law for the decay probabilities $P_\eta(t)$. *If the exponential law*

holds, the total decay rate R and the inverse lifetime are the same[1]. However, the inverse lifetime $1/\tau$ of (2.5) and the width Γ of the Breit-Wigner energy distribution (2.2) are conceptually and experimentally different quantities. The width Γ is determined experimentally from the fit of the scattering data to the Lorentzian lineshape (2.2) of a resonance scattering experiment (2.1). The lifetime $\tau = 1/R$ is determined experimentally from a fit to the exponential time dependence of the counting rate of the decay products (2.5).

Nevertheless, one often calls the calculated quantity $\Gamma^{\rm calc} = \hbar R = \hbar/\tau$ with τ measured by (2.5), also the width of the decaying particle, and the $\hbar R_\eta \equiv \Gamma_\eta$ are usually called the partial widths.

Within traditional quantum mechanics (using the Hilbert space axiom) one cannot derive the equality of Γ in (2.2) and $\hbar R = \hbar/\tau$ in (2.5) or any other relationship between the width Γ and the inverse lifetime $\frac{\hbar}{\tau}$. One can also not derive the exponential law (2.5) from (2.6) using (2.7), because there is no vector $\phi^D(t)$ in Hilbert space for which the right hand side of (2.7) with (2.6) would have an exponential time dependence.

The idea that τ and $\hbar/\Gamma$ are the same or are at least approximately equal is based on the Weisskopf-Wigner approximation methods [1,3]. In the monograph [3] one considers a prepared state ϕ^D, which has a Breit-Wigner energy wave function (2.2). Then one calculates the probability $\mathcal{P}_D(t)$ for finding this state ϕ^D at a time t and obtains (section 8.2 of [3]):

$$\mathcal{P}_D(t) = e^{-\Gamma t/\hbar} + \Gamma \times \text{small terms}\,. \tag{2.9}$$

Neglecting the small terms one can conclude for the average lifetime of (2.8) that $\tau \approx \frac{\hbar}{\Gamma}$.

Using these kind of approximate methods a number of important empirical notions have been introduced over the years: decaying Gamow vectors [5] with complex energy eigenvalues, Breit-Wigner (Lorentzian) resonance amplitudes [6], the Lippmann-Schwinger in- and out-plane wave states [7], the

[1]The average lifetime is defined as the average value of the time intervals Δt_i that any one of the particles D survives:

$$\frac{1}{N_D}\sum_i N_D(\Delta t_i)\Delta t_i \approx \frac{1}{N_D}\int N_D(t)dt = \tau \tag{2.8}$$

where $N_D = N_D(0) = N_D(t) + \sum_\eta N_\eta(t)$ is the number of decaying particles at $t = 0$, $N_D(t)$ is the number of decaying particles at time t, and $N_\eta(t)$, $\eta = \eta_1\,,\eta_2\,,\cdots$, are the numbers of decay products η at time t. If the exponential law $\mathcal{P}_D(t) \approx \frac{N_D(t)}{N_D} = e^{-Rt}$ holds, and only if the exponential law holds, is the average lifetime given by $\frac{1}{R} = \tau$.

analytically continued S-matrix and its resonance poles [8]. While these methods provided a means to perform calculations leading to results which agreed with the experiments to a satisfactory degree of accuracy, they also led to many puzzles and contradictions: complex energy versus the self-adjointness of the Hamiltonian; exponential decay law versus the deviation from the exponential time evolution for any vector in the Hilbert space; [9] exponential catastrophe versus the unitary (reversible) time evolution [10]; and causality versus the semi-boundedness of the Hamiltonian [11].

The conclusion is that an exact, mathematically consistent theory of resonance scattering and decay does not exist [12]. Neither Breit-Wigner resonances nor exponentially decaying Gamow states are possible within the frame of the traditional Hilbert space axiom of quantum mechanics. Furthermore, the decay of excited atoms [13] and of elementary particles [14] is a time asymmetric (sometimes also called irreversible) process. But time evolution in Hilbert space is always time symmetric since the Schroedinger and Heisenberg equations lead to the unitary ("reversible") time evolution (Stone-von Neumann theorem when solved under the Hilbert space boundary condition) [15].

3 Modification of one traditional axiom of quantum theory

Therefore a modification of the Hilbert space theory of quantum physics is needed. This modification started with Dirac's kets for which Schwartz created his theory of distributions, and which Gel'fand, Maurin and their schools generalized to the Rigged Hilbert Space theory, to prove the general Dirac basis vector expansion as the Nuclear Spectral Theorem [16, 17]. The quantum theory of resonances and decay and of asymmetric time evolution requires particular versions of Rigged Hilbert Spaces, the pair in which the base spaces are Hardy spaces [18–20].

Dirac kets $|E\rangle$ and Dirac's δ-distribution are well accepted entities that lie beyond the mathematics of the Hilbert space. Only few physics books give their mathematical definition [4]: the $|E\rangle$ are defined as functionals over the abstract Schwartz space and $\delta(E - E_0)$ are functionals on the space of smooth rapidly decreasing functions (Schwartz space functions) $S|_{\mathbb{R}_+}$. The energy wave functions are Schwartz space functions: $\phi(E) = \langle E|\phi\rangle \in S|_{\mathbb{R}_+}$.

This means Dirac kets and Dirac's formulation of quantum mechanics requires a Gel'fand triplet of spaces [19]

$$\Phi \subset \mathcal{H} \subset \Phi^\times \quad \text{with} \quad \phi \in \Phi, \quad |E\rangle \in \Phi^\times\,, \tag{3.1}$$

where one chooses for Φ the abstract Schwartz space.

The same is expected of the other generalized vectors of scattering and decay theory, like the Lippmann-Schwinger kets $|E^-\rangle$ and $|E^+\rangle$. But these cannot be functionals over the Schwartz space because of the infinitesimal $\mp i\epsilon$ in the Lippmann-Schwinger equation, which is a means of formulating two distinct (outgoing and incoming) boundary conditions. To define kets which allow analytic continuation into the complex energy plane, and therewith the formulation of outgoing and incoming boundary conditions, requires that the energy wave functions: $\phi^+(E) = \langle^+E|\phi^+\rangle$ and $\psi^-(E) = \langle^-E|\psi^-\rangle$ be "better" than Schwartz space functions. They must be functions that can be analytically continued into the upper (for $\langle^-E|\psi^-\rangle$) and lower (for $\langle^+E|\phi^+\rangle$) complex energy plane (of the second Riemann sheet of the analytically continued S-matrix where the resonance poles are located).

Therefore we make the new Hardy space hypothesis:

The prepared in-states ϕ^+ (experimentally given by the preparation apparatus) are described by

$$\{\phi^+\} = \Phi_- \subset \mathcal{H} \subset \Phi_-^\times\,, \qquad |E^+\rangle \in \Phi_-^\times\,, \tag{3.2}$$

and the detected out-states, or precisely out-observables ψ^- (because they are experimentally given by the detector) are described by

$$\{\psi^-\} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times\,, \qquad |E^-\rangle \in \Phi_+^\times\,. \tag{3.3}$$

The two spaces Φ_+ and Φ_- are two different Hardy sub-spaces (analytic in the upper and lower energy semi-plane respectively) which are dense in the same Hilbert space $\mathcal{H}$. This means the energy wavefunctions $\phi^+(E) = \langle^+E|\phi^+\rangle$ of the in-state ϕ^+ and the energy wavefunctions $\psi^-(E) = \langle^-E|\psi^-\rangle$ of the out-observable ψ^- are those Schwartz space functions of E which can be analytically continued into the lower complex semi-plane for $\phi^+(E)$ and into the upper complex semi-plane for $\psi^-(E)$, and which vanish rapidly enough at the infinite semicircle; they are in the intersections of the Schwartz space $\mathcal{S}$ and the Hardy spaces $\mathcal{H}^2_\mp$ [19, 20]:

$$\langle^+E|\phi^+\rangle \equiv \phi^+(E) \in \mathcal{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_+}\,, \tag{3.2a}$$

$$\langle^-E|\psi^-\rangle \equiv \psi^-(E) \in \mathcal{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_+} \Rightarrow \overline{\langle^-E|\psi^-\rangle} \equiv \langle\psi^-|E^-\rangle \in \mathcal{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_+}\,. \tag{3.3a}$$

The values of the Hardy function $\phi^+(z)$ and $\overline{\psi^-(E)}$ in the lower complex semi-plane second sheet $\mathbb{C}_-$ including the negative real axis are completely determined from the values at the positive real axis $\mathbb{R}_+$ (the spectrum of the Hamiltonian).

The new hypothesis (3.2) (3.3) accounts for the fact that the states ϕ^+ and the observable $|\psi^-\rangle\langle\psi^-|$, which are experimentally defined by different parts of the experiment ($\phi^+ \in \Phi_-$ by the preparation apparatus for the in-state and $\psi^- \in \Phi_+$ by the detector), are also represented mathematically by different "parts" of the Hilbert space $\mathcal{H}$. Different "parts" here means different *dense* subspaces of $\mathcal{H}$[2]. This means, as long as one considers only algebraic notion, leaving notions like convergence or completeness aside, then the Hilbert space axiom $\{\phi^+\} = \{\psi^-\} = \mathcal{H}$ and the Schwartz space axiom $\{\phi^+\} = \{\psi^-\} = \Phi \subset \mathcal{H}$ and the new hypothesis (3.2, 3.3) "are all same".

By direct observation it is also difficult to distinguish between the hypothesis (3.1) and the new hypothesis (3.2, 3.3). The assumption (3.1) would mean that the detector efficiency $|\psi^-(E)|^2$ and the energy distribution of the beam $|\phi^+(E)|^2$ are described by smooth functions, and the new hypothesis (3.2, 3.3) means that the detector is described only by smooth functions $\{\psi^-(E)\}$ that can be analytically continued into the upper complex energy semi-plane $\mathbb{C}_+$, and the preparation apparatus is described only by those smooth functions $\phi^+(E)$ which can be analytically continued into the lower complex semi-plane $\mathbb{C}_-$. Mathematically this analyticity requirement makes a significant difference, and if one takes the Fourier transform, one obtains from the analyticity hypothesis of $\phi^+(E)$ and $\psi^-(E)$ the time asymmetry $t \geq 0$ and causality.

Also, since $\Phi_\mp \subset \Phi$ and therefore $\Phi_\mp^\times \supset \Phi^\times$, there are more generalized eigenvectors (kets) under the hypothesis (3.2, 3.3) than the Dirac kets $|E\rangle$ of (3.1). In particular there are the analytic continuation of the bras $\langle^- E|$, $E \in \mathbb{R}_+$, into the upper complex plane $\mathbb{C}_+$, and therefore of the $|E^-\rangle$ into the lower complex plane, second sheet $\mathbb{C}_-$ and the

$$|z^-\rangle \in \Phi_+^\times \qquad \text{are defined for all } z \in \mathbb{C}_- \,, \tag{3.4}$$

as long as these $z \in \mathbb{C}_-$ are not singularities of the analytically continued S-matrix $S(z)$. Similarly the $\langle^+ E|\phi^+\rangle$, $E \in \mathbb{R}_+$, can be analytically continued into the lower complex plane, second sheet, and bras $\langle^+ E|$ are thus defined

[2]like the rational numbers being a dense subset of the real numbers

also for complex $z \in \mathbb{C}_-$

$$\langle^+ z| \in \Phi_-^\times \qquad \text{for } z \in \mathbb{C}_- \tag{3.5}$$

as long as these $z \in \mathbb{C}_-$ are not singularities of the analytically continued $S(z)$. Here $\mathbb{C}_-$ refers of the lower complex semi-plane of the second sheet of the Riemann surface of the S-matrix.

In addition to the continuum of kets $|z^-\rangle \in \Phi_+^\times$ obtained by analytic continuation from the $|E^-\rangle$ there are also the kets $|z_R^-\rangle \in \Phi_+^\times$ which correspond to the discrete sets of first order poles at $z_R = E_R - i\Gamma/2$ of the S-matrix. These kets we shall call Gamow vectors; they are the central concepts of the theory of resonance scattering and decay based on the new axiom (3.2), (3.3). We exclude from our discussion here higher order poles of the S-matrix (which are discussed in [21]) and other singularities.

The axiom (3.2), (3.3) has been postulated as a replacement of the Hilbert space axiom in order to accommodate these new vectors $|E^-\rangle \in \Phi_+^\times$, $|E^+\rangle \in \Phi_-^\times$ and the Gamow vectors [5], $|z_R^-\rangle \in \Phi_+^\times$. This replacement of (3.1) by (3.2), (3.3) is the only modification of the traditional formulation [4], that uses Hilbert space and Dirac kets of (3.1). All other axioms of quantum theory remain intact, but these other axioms are extended to the new objects like the Gamow kets $|z_R^-\rangle$ and the kets $|E - i\epsilon^-\rangle$, the bras $\langle^+ E - i\epsilon|$ and their analytic continuations $|z^-\rangle$, $\langle^+ z|$, which we shall call Lippmann-Schwinger kets [7] because of their analogy. In particular, the fundamental Born probabilities of Axiom (2.6) for an observable $|\psi_\eta^-(t)\rangle\langle\psi_\eta^-(t)|$ in the state ϕ^+, $|\langle\psi_\eta^-(t)|\phi^+\rangle|^2$ are extended to Gamow states: $\phi^+ \to |z_R^-\rangle$. Thus the probability to detect the observable $|\psi_-^\eta(t)\rangle\langle\psi_-^\eta(t)|$ in the Gamow state $|z_R^-\rangle$ is in analogy to (2.6), with $\phi^D \to |z_R^-\rangle$, given by:

$$\mathcal{P}_\eta(t) \sim \mathrm{Tr}\left(|\psi_\eta^-(t)\rangle\langle\psi_\eta^-(t)|\, |z_R^-\rangle\langle^- z_R|\right) = |\langle\psi_\eta^-(t)|z_R^-\rangle|^2 \sim |\langle\psi_\eta^-(t)|\phi^G\rangle|^2 \,. \tag{3.6}$$

These new probabilities like (3.6) are mathematically well defined quantities, (i.e., the value of the functional $|z_R^-\rangle \in \Phi_+^\times$ at the point $\psi^- \in \Phi_+$) and represent Born probabilities for the decay products η at time t in the "Gamow state" $\phi^G \sim |z_R^-\rangle$.

The Gamow vectors $|z_R^-\rangle$ are the generalized eigenvectors or eigenkets of the self-adjoint Hamiltonian H with complex eigenvalue z_R, they are associated to the first order pole of the S-matrix. They have a Breit-Wigner

energy distribution (2.2) and an exponential time evolution. We give a brief sketch to show how this follows from the new axiom (3.2, 3.3), for details see [18, 22].

Since the Hardy space triples (3.3) and (3.2) are Rigged Hilbert spaces, one has for every $\phi^{\pm} \in \Phi_{\mp}$ the Dirac basis vector expansion (nuclear spectral theorem)

$$\phi^{\pm} = \sum_{j,\eta} \int_0^{\infty} dE |E, j, \cdots, \eta^{\pm}\rangle\langle^{\pm}E, j, \cdots, \eta|\phi^{\pm}\rangle \, . \tag{3.7}$$

Under the Hardy space hypothesis (3.2), (3.3) the contour of integration in (3.7) can be deformed into the complex semi-plane C_- (in (3.7), $\cdots$ are additional quantum numbers like angular momentum j , j_3 etc, and η denotes the channel or particle species label.)

The starting point for the derivation of the Gamow ket is the Born probability amplitude for the observable $\psi^- \in \Phi_+$ in the state $\phi^+ \in \Phi_-$, i.e. the S-matrix element

$$(\psi^{\rm out}, S\phi^{\rm in}) = (\psi^-_{\eta}, \phi^+_{\eta^0}) = \sum_j \int_0^{\infty} dE \langle\psi^-_{\eta}|E^-\rangle S_j^{\eta\,\eta_0}(E)\langle^- E|\phi^+_{\eta^0}\rangle. \tag{3.8}$$

The right hand side of (3.8) is obtained from (3.7) with

$$S_j^{\eta,\eta^0}(E) \equiv \langle^- E, j, \eta|E, j\eta^0 \cdots {}^+\rangle = 2ia_j^{\eta}(E) \quad (\eta \neq \eta_0) \, . \tag{3.9}$$

and using (rotation) invariance, for details see [18, 22].

Under the Hardy class hypothesis (3.2, 3.3) for the wavefunctions $\langle E^-|\psi^-\rangle$ and $\langle E^+|\psi^+\rangle$, the integral in (3.8) can be carried out over any contour in the lower half complex plane 2nd sheet of the S-matrix as long as it avoids singularities. If this integration is performed on a contour that extends beyond the position z_R of the resonance pole, then one has to make a circle around it.

To the integral around this circle only the pole term $\frac{R_{-1}}{z-z_R}$ of the S-matrix

$$S_j^{\eta,\eta'}(z) = \frac{R_{-1}}{z - z_R} + R_0 + R_1(z - z_R) + \cdots \tag{3.10}$$

contributes and this contribution is according to the Cauchy integral formula given by

$$-2\pi i R_{-1}\langle\psi^-|z_R^-\rangle\langle^+ z_R|\phi^+\rangle \, .$$

Since ψ^- in (3.8) is arbitrary, we conclude that $|z_R^-\rangle$ is a functional on $\Phi_+ = \{\psi^-\}$. The value of this functional at $\psi^- \in \Phi_+$ is given by

$$\langle\psi^-|z_R^-\rangle = \frac{i}{2\pi}\oint dz \frac{\langle\psi^-|z^-\rangle}{z - z_R} = \frac{i}{2\pi}\int_{-\infty_{II}}^{+\infty} dE \frac{\langle\psi^-|E^-\rangle}{E - z_R}\,, \tag{3.11}$$

where $z_R = E_R - i\Gamma_R/2$ is the position of the resonance pole of the analytic S-matrix. The first equality in (3.11) is the Cauchy integral formula for the function $\langle\psi^-|z^-\rangle$ and the second equality is the Titchmarsh theorem for Hardy functions $\langle\psi^-|E^-\rangle$. The integral extends along the real axis in the 2nd sheet as indicated by $-\infty_{II}$.

This equation (3.11) one also writes in Dirac notation as an equation between functionals (omitting the arbitrary $\psi^- \in \Phi_+$); one defines a "normalized" Gamow ket $\phi_j^G \in \Phi_+^\times$:

$$\phi_j^G = \frac{\sqrt{2\pi\Gamma}}{f}|z_R, j, \cdots^-\rangle = \frac{i\sqrt{2\pi\Gamma}}{2\pi f}\int_{-\infty_{II}}^{+\infty} dE \frac{|E_j, \cdots^-\rangle}{E - z_R}\,, \tag{3.12}$$

where f is a suitable "normalization" factor for the Gamow vector ϕ_j^G. Gamow kets $|z_R, j, \cdots^-\rangle$ are thus the singular points of the analytically continued Lippmann-Schwinger kets $|E, j, \cdots^-\rangle$ associated to the state given by the pole at z_R.

For the vector $\phi_j^G \in \Phi^\times$ defined in (3.12) and *only* if the integral in (3.12) extends to $-\infty_{II}$ one can derive (using the Hardy hypothesis (3.3)) that

$$\langle H\psi_\eta^-|\phi^G\rangle \equiv \langle\psi_\eta^-|H^\times|\phi^G\rangle = (E_R - i\Gamma/2)\langle\psi_\eta^-|\phi^G\rangle \text{ for all } \psi_\eta^- \in \Phi_+ \tag{3.13}$$

where $H = H_0 + V$ is self-adjoint (and semi-bounded [3]). The result (3.13) justifies the notation

$$\phi_j^G = \sqrt{2\pi\Gamma}|E_R - i\Gamma/2, \cdots^-\rangle.$$

In Dirac's notation the arbitrary $\psi_j^- \in \Phi_+$ is omitted and (3.13) is written as

$$H^\times|E_R - i\Gamma/2, j, \cdots^-\rangle = (E_R - i\Gamma/2)\ |E_R - i\Gamma/2, j, \cdots^-\rangle\,. \tag{3.14}$$

[3] The spectrum of a Hamiltonian is always bounded from below("stability of matter"); in (3.8) we chose this bound to be zero and ignored bound state poles which are not relevant for our discussion here

The operator $H^\times$ is uniquely defined by:

$$\langle H\psi^- | F^- \rangle = \langle \psi^- | H^\times | F^- \rangle \quad \text{for every} \quad F^- \in \Phi_+^\times\,, \quad \psi^- \in \Phi_+\,;$$

it is the extension of the operator $H^\dagger = H$ to $\Phi_+^\times$. For the Gel'fand triplet over Schwartz space (3.1), self-adjoint operators H have only real generalized eigenvalues. For other Gel'fand triplets, like the Hardy triplets (3.2) and (3.3), the generalized eigenvalues of self-adjoint H can also be complex. (Dirac also omitted the $^\times$ of $H^\times$ when he wrote the eigenvalue equation (3.14) for real generalized eigenvalues.)

An important result that one can derive for the vector ϕ_j^G defined by (3.11) or (3.12) is [18, 22]

$$\begin{aligned}&\langle \psi_\eta^-(t) | \phi_j^G \rangle \sim \langle e^{iHt/\hbar}\psi_\eta^- | E_R - i\Gamma/2^- \rangle \\ &\equiv \langle \psi_\eta^- | e^{-iH^\times t/\hbar} | E_R - i\Gamma/2^- \rangle = e^{-iEt/\hbar} e^{-\frac{\Gamma}{2}t/\hbar} \langle \psi_\eta^- | E_R - i\Gamma/2^- \rangle \\ &\qquad\qquad\qquad \forall \quad \psi_\eta^- \in \Phi_+ \quad \text{and} \ \boxed{\text{for } t \geq 0 \text{ only}}\,. \end{aligned} \tag{3.15}$$

4 Quantum mechanical time asymmetry

The importance of the result (3.15) is that it can be obtained from (3.11) with $\psi^- \to \psi_\eta^-(t)$ only for $t \geq 0$, since $\psi_\eta^-(t) = e^{iHt/\hbar}\psi_\eta^-$ is not an element of Φ_+ for $t < 0$. The result (3.15) written as a functional equation for the Gamow ket is:

$$\phi_j^G(t) = e^{-iH^\times t/\hbar}\phi_j^G = e^{-iEt/\hbar} e^{-(\Gamma/2)t/\hbar}\phi_j^G \quad \text{for } t \geq 0 \quad \text{only}\,. \tag{4.1}$$

This means that the Gamow ket (3.12) defined as a functional (3.11), (3.12), with Breit-Wigner energy distribution $\frac{1}{E-z_R} = \frac{1}{E-(E_R - i\Gamma/2)}$ has a time evolution (3.15) for $t \geq 0$ only. The Gamow kets (4.1) are defined as functionals for $t \geq 0$ only. The time evolution operators form a semigroup $U^\times(t) = e^{-iH^\times t/\hbar}$, $t \geq 0$, and *not* a unitary group like the time evolution in the Hilbert space given by: $U^\dagger(t) = e^{-iHt/\hbar}$, $-\infty < t < \infty$, with the inverse $(U^\dagger(t))^{-1} = U^\dagger(-t)$. The operator $U^\times(t) = e^{-iH^\times t/\hbar}$ with $0 \leq t < \infty$ has no inverse, hence $\{U^\times(t) | 0 \leq t < \infty\}$ form a semigroup.

Since $\langle \psi_\eta^- | \phi^G(t) \rangle$ represents according to (3.6) the probability amplitude for the decay product η (described by ψ_η^-) in the state $\phi^G(t)$ we have derived the exponential law:

$$|\langle \psi_\eta^-(t) | \phi^G \rangle|^2 = |\langle \psi_\eta^- | \phi^G(t) \rangle|^2 = e^{-\Gamma t/\hbar} |\langle \psi_\eta^- | \phi_j^G(0) \rangle|^2, \quad \text{for} \quad t \geq 0\,. \tag{4.2}$$

This is the exponential law which leads to the exponential decay rate formula (2.5) of $R_\eta(t)$, by which the lifetime is measured iff $\tau = \hbar/\Gamma$, where $\Gamma = -2\mathrm{Im}(z_R)$ is the width of the Breit-Wigner resonance (2.2).

To summarize, the pole of the j-th partial S-matrix $S_j(z)$ in the lower half complex energy semi-plane second sheet at $z_R = E_R - i\Gamma/2$, the first term of (3.10), defines the resonance with resonance energy E_R and resonance width Γ. From this resonance pole one obtains a Gamow vector by (3.11) and (3.12). This Gamow vector is an eigenket of the Hamiltonian fulfilling (3.13), (3.14). It describes an exponentially decaying state with a lifetime precisely given by $\tau = \frac{\hbar}{\Gamma}$.

This Gamow vector, defined by (3.11), (3.12) from the S-matrix pole, has an energy distribution given by the "idealized" Breit-Wigner (Lorentzian)

$$\frac{1}{E - (E_R - i\Gamma/2)} \qquad -\infty < E < \infty\,, \tag{4.3}$$

that extends over the entire real axis, whereas the (Hilbert space) spectrum of $\bar{H}$ is $0 \leq E < \infty$.

The Gamow vector $|z_R^-\rangle \in \Phi_+^\times$ is a generalized vector (ket) on the Hardy space Φ_+, which is isomorphic (algebraically and topologically) to the space of wave functions $\{\langle^+E|\phi^+\rangle\} = \mathcal{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_+} = \{\langle\psi^-|E^-\rangle = \overline{\langle^-E|\psi^-\rangle}\}$ analytic in the lower complex semi-plane. The Hardy function $\langle^+z|\phi^+\rangle$ (in particular the $\langle^+E|\phi^+$ for $E \in \mathbb{R}_-$) is already completely determined by its values for $E \in \mathbb{R}_+$ [24]. The Hardy space vectors $\phi^+ \in \Phi_-$ (*i.e.* $\langle^+E|\phi^+\rangle \in \mathcal{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_+}$) are represented by smooth well behaved functions $\langle^+E|\phi^+\rangle$ in the spectral resolution

$$\phi^+ = \int_0^\infty dE'\,|E'^+\rangle\langle^+E'|\phi^+\rangle.$$

However the Gamow ket $\phi^G \sim |z_R^-\rangle$ cannot be represented in this way, in particular

$$\phi^G \text{ is not } = \int_0^\infty dE'|E'^+\rangle \frac{1}{E' - (E_R - i\frac{\Gamma}{2})}.$$

But $\langle^+E|\phi^G\rangle$ is an intricate singular expression [25]. Nevertheless, ϕ^G has the representation (3.11), (3.12) by a smooth function (4.3) on the whole real line $\mathbb{R}$, with $-\infty < E < +\infty$. It is this complicated relationship between the exponential time dependence (4.1) for the Gamow vector and the

Breit-Wigner energy distribution (4.3) on $\mathbb{R}$, but not on $\mathbb{R}_+$ which made the uncovering of $\tau = \hbar/\Gamma$ as an exact relation so difficult.

The Hardy space axiom (3.2), (3.3) provides a unified theory of resonances and decay. This was the purpose for which the Hardy space hypothesis was introduced [18], it relates quantum decay with resonance scattering. Further, the Hardy space admits Gamow state vectors with an exponential time evolution of lifetime τ (which cannot exist in a Hilbert space [9]) and relates them to vectors with an idealized Breit-Wigner energy distribution (4.3) of width Γ. And it led to the lifetime-width relation $\tau = \hbar/\Gamma$, which physicists always desired as an exact equality. This equality has recently been verified with an accuracy that exceeds the accuracy expected by the Weisskopf-Wigner approximation [23].

For the unification of resonance and decay phenomena, the Hardy axiom should be welcome. However, there is another conclusion drawn from hypothesis (3.2) (3.3) which is expressed in (3.15) and (4.1) by the time asymmetry $t \geq 0$, and this is not an easily acceptable feature. It is a mathematical consequence of the boundary conditions $\psi^- \in \Phi_+$, $\phi^+ \in \Phi_- \subset \Phi_+^\times$, $\phi^G \in \Phi_+^\times$ for the time symmetric dynamical equation (the Heisenberg equation for ψ^- and the Schrodinger equation for ϕ^+ and ϕ^G) inherent in the hypothesis (3.2) (3.3). Whereas the solutions of the dynamical equations under the Hilbert space boundary conditions and under (3.1) are given (according to the Stone-von Neumann theorem [15]) by the unitary group

$$\phi(t) = e^{-i\bar{H}t/\hbar}\phi, \qquad -\infty < t < +\infty \qquad \text{(for the Schrodinger equation)} \tag{4.4}$$

and

$$\psi(t) = e^{i\bar{H}t/\hbar}\psi, \qquad -\infty < t < +\infty \qquad \text{(for Heisenberg equation),} \tag{4.5}$$

the solutions of the same dynamical equation under the Hardy space boundary conditions (3.2) (3.3) are given (according to the Paley-Wiener theorem [26]) by the semigroups:

$$\phi^+(t) = e^{-iH^\times t/\hbar}\phi^+, \qquad 0 \leq t < +\infty \qquad \text{(for the Schrodinger equation)} \tag{4.6}$$

and

$$\psi^-(t) = e^{iHt/\hbar}\psi^-, \qquad 0 \leq t < +\infty \qquad \text{(for the Heisenberg equation).} \tag{4.7}$$

The results (4.6) (4.7) mean that the time evolution operator $e^{iHt/\hbar}$ is a continuous operator (with respect to the topology in $\Phi_{\mp}$) only for $t \geq 0$ and therefore the time evolution operator $e^{-iH^{\times}t/\hbar}$ in $\Phi_{\mp}^{\times}$ is defined only for $t \geq 0$. In physical terms this means that only for $t \geq 0$ will the Born probabilities (3.6) always be finite. The restriction of the time evolution to the semigroup $U(t) = e^{iHt/\hbar}$ ($t \geq 0$) for observables and to $U^{\times}(t) = e^{-iH^{\times}t/\hbar}$ ($t \geq 0$) for states, is the way how the Hardy axiom avoids infinite Born probabilities (the "exponential catastrophe") even though it admits such vectors like Gamow kets ϕ^G to represent physical entities, like exponentially decaying states.

For the interpretation in terms of states ϕ^+ and observables $\psi_\eta^-(t)$, time asymmetry $t \geq 0$ means that the state ϕ^+ must be prepared first, at a time $t_0 = 0$, before the decay products represented by $|\psi_\eta^-(t) >< \psi_\eta^-(t)|$ can be registered by the η-detector at time $(t - t_0) > 0$. This is a reasonable *condition of causality.*

All these features are not contained in the traditional (Hilbert space) quantum mechanics where the Born probabilities $|(\psi(t), \phi)|^2$ are defined for all times $-\infty < t < +\infty$. But the Hilbert space probabilities are not without pathologies: One can show that they must be different from zero for all time, unless they are identically zero [11], i.e., there is no decay of a state which had been prepared at a finite time t_0 and before which time the probabilities were zero. Further there exists no state vector ϕ with exponential Born probabilities [9].

5 Conclusion

The irreversible nature of quantum decay which can be understood as a consequence of the time asymmetry (4.6) (4.7), has been mentioned in textbooks [13, 14] and lecture notes [27, 28].

The time t_0 before which "the state is defined completely by the preparation" has already been mentioned by Feynmann [29]; and Gell-Mann and Hartle [28] applied this idea to the probabilities of history (for the expanding universe considered as a closed quantum system). They did not derive (4.7) but restricted by fiat the time in $e^{iH(t-t_0)}$ to $t \geq t_0 =$ big-bang. Our universe considered as a quantum physical system (one specimen of an ensemble of many worlds) could be in states $\rho(t) = e^{-iH(t-t_0)}\rho(t_0)e^{iH(t-t_0)}$ only for $t \geq t_0 \equiv t_{\text{big-bang}}$. The big-bang time would thus provide an example for the semigroup time t_0. Other systems where one could get an indication of

the existence of the semigroup time t_0 is the slow (weak) decay of quasi-stable particles produced by a strong interactions [30].

In general, it is difficult to recognize the quantum mechanical beginning of time t_0 since in micro physics one studies a large ensemble of (identical) quantum systems prepared at a collection of numerous times. This makes it impossible to pin-point a particular time t_0 at which the quantum state has been prepared.

For this reason the existence of a quantum mechanical beginning time t_0 remained obscure for long. However, recently, experiments with single ions [31] changed this situation. We shall discuss in the subsequent publication II how these experiments demonstrate the beginning of time for a quantum state.

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